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LETTER TO THE EDITOR

Special relations for continuous and discrete Painlevé equations

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Abstract. We investigate a relation between solutions of the Painlevé II equation corresponding to values of the parameter of P_{II} which cannot be connected through a Schlesinger transformation, which was first derived by Gambier. Here we present the discrete analogue of this relation, relating the discrete P_{II} to the alternate d-P_{II}. The latter turns out to be a consequence of a quadratic relation existing between two different families of solutions of P_V. A q-discrete analogue of the latter relation is also presented.

What is amazing about Painlevé equations is not the extreme richness of their special properties but rather the fact that new relations are regularly being discovered more than a century after their derivation [1]. One possible reason for the renewed interest in the Painlevé equations is the discovery of their discrete analogues a mere decade ago [2]. The term discrete Painlevé (d-$\mathbb{P}$) has been coined to indicate a non-autonomous, integrable mapping which, at the continuous limit, goes over to a Painlevé equation. Soon after the d-$\mathbb{P}$'s made their appearance it was remarked that d-$\mathbb{P}$'s appear in, essentially, two different forms, additive (i.e. as difference equations) and multiplicative (i.e. as q-equations) [3]. In the former the independent variable n enters linearly while in the latter it enters exponentially. Moreover, a large class of difference d-$\mathbb{P}$'s are just contiguity relations of the continous ones [4]. Thus some of the properties of the (continuous) Painlevé equations are expected to have direct consequences on the discrete case.

One of the remarkable properties of Painlevé equations is that while they introduce new special functions, new 'transcendents', they also possess some solutions that can be expressed in terms of elementary functions [5]. These solutions fall into two classes: solutions which are rational in the independent variable, and solutions which are expressed in terms of the classical special functions. Since the latter are the solutions of linear equations, this second kind of solutions is referred to as the 'linearizable' case. These special solutions exist only for very particular values of the parameters of the Painlevé equations. Moreover, the intersection of the conditions for existence of both linearizable and rational solutions is sometimes empty. Yet, for some Painlevé equations there exists a relation between the solutions corresponding to values of the parameters associated with rational and with linearizable solutions. The simplest, and best known, of these relations is the one for Painlevé II. It was first discovered by Gambier [6] and has been recently rediscovered by several authors [7] (including those of the present

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paper). However, relations of this type do exist for other $\mathbb{P}$'s as well and moreover, as we shall show in what follows, these relations are not restricted to the continuous case but appear equally well for d-$\mathbb{P}$'s.

Let us start with the (continuous) Painlevé II equation and a brief recollection of its properties. In what follows the prime ($'$) denotes a derivative with respect to the independent variable, i.e. $v' = \mathrm{d}v/\mathrm{d}t$. The $\mathrm{P_{II}}$ equation which reads

$$v'' = 2v^3 + tv + \mu \tag{1}$$

with $\mu = -2\alpha - \frac{1}{2}$, has special solutions for μ integer (rational solutions) and μ half-integer (linearizable case). Moreover, it is related to another equation of the Painlevé/Gambier classification through the Miura transformation [8]:

$$2\alpha w = v' + v^2 + t/2 \tag{2a}$$

$$v = \frac{w' + 1}{2w}. \tag{2b}$$

The equation for w, traditionally referred to as $\mathrm{P_{34}}$, is usually, as in Ince's monograph [8], presented in the form:

$$w'' = \frac{(w')^2}{2w} + 4\alpha w^2 - tw - \frac{1}{2w}. \tag{3}$$

What happens to $\mathrm{P_{34}}$ when α=0? Clearly from (3) we have the equation

$$w'' = \frac{(w')^2}{2w} - tw - \frac{1}{2w} \tag{4}$$

This equation does *not* introduce a new transcendent. As a matter of fact equation (4), and also the equation $w'' = (w')^2/2w - f(t)w - 1/2\omega$ for arbitrary $f(t)$, can be reduced to the canonical form

$$w'' = \frac{(w')^2}{2w} - \frac{1}{2w}. \tag{5}$$

This is just equation XXXII of the Painlevé/Gambier list. The latter has the solution $w = at^2 + bt + c$ with the constraint $b^2 - 4ac = 1$, while the equation with an arbitrary $f(t)$ is solved by $w = A(t)B(t)$ with A, B two solutions of the equation $X'' = -fX/2$, with the constraint that their Wronskian $A'B - B'A$ be ± 1.

However, it is clear from (2) that the Miura transformation becomes meaningless when $\alpha = 0$. In order to remedy this we rescale w, introducing $W = \alpha w$, whereupon $\mathrm{P_{34}}$ becomes

$$W'' = \frac{(W')^2}{2W} + 4W^2 - tW - \frac{\alpha^2}{2W} \tag{6}$$

and the Miura relating it to $\mathrm{P_{II}}$ is

$$2W = v' + v^2 + t/2 \tag{7a}$$

$$v = \frac{W' + \alpha}{2W}. \tag{7b}$$

In the normalization of (6) and (7) it is straightforward to take the $\alpha = 0$ limit. The equation we obtain in this case is, up to a scaling, equation XX of the Painlevé/Gambier list. The method of integrating the latter is through a quadratic transformation $u^2 = W$ which reduces it to a $\mathrm{P_{II}}$ with $\mu = 0$, i.e.

$$u'' = 2u^3 - \frac{tu}{2}. \tag{8}$$

With these elements at hand it is straightforward to establish the relation found by Gambier. We start from a $\mathrm{P_{II}}$ with $\mu = -\frac{1}{2}$, i.e. a value of the parameter corresponding to the linearizable case. Next, we perform the Miura, in the renormalised form, i.e. $2W = v' + v^2 + t/2$. Since $\mu = -\frac{1}{2}$ corresponds to $\alpha = 0$ the resulting equation for W is not P_{34} but rather P_{20} and by taking $u^2 = W$ we find for u a P_{II} with $\mu = 0$, equation (8). This value of the parameter corresponds to the rational-solution case. Thus, with this chain of transformations we have obtained a relation between solutions of P_{II} equation (1) with $\mu = -\frac{1}{2}$, to solutions of a P_{II} with $\mu = 0$ in the slightly non-canonical form (8) given above. The Miura reads

$$2u^2 = v' + v^2 + t/2 \tag{9a}$$

$$v = \frac{u'}{u}. \tag{9b}$$

This is a remarkable property because the solutions for these two cases cannot be related through the usual Schlesinger transformations which relate solutions with $\Delta\mu = \pm 1$. We must point out here that, once the basic relation between solutions of the $\mu = 0$ and $\mu = -\frac{1}{2}$ equations is obtained, Schlesinger transformations can be brought to play in order to relate solutions between any integer μ and any half-integer μ. From the form of (9) it is clear that while u determines completely v, if we give the latter we can determine u only up to a $\pm$ sign. This is due to the fact that both $\pm u$ satisfy P_{II} with $\mu = 0$, equation (8).

Note that the latter fact has an interesting offshoot. Suppose we start from the solution v of P_{II}, equation (1), for some integer μ and apply successive Schlesinger transformations in order to relate it to the solution at $\mu = 0$. We then change the sign of the latter solution and using the inverse Schlesinger transformations we construct another solution $\tilde{v}$ of P_{II} for the same integer μ. This constitutes a duality between the solutions of P_{II} for any integer μ (these being the values of the parameter where rational solutions exist). Note that the (unique) rational solution for each value of μ is self-dual, since it goes over to the zero solution of P_{II} with $\mu = 0$. By following the chain of transformations we have just described, it is possible to relate explicitly the two solutions. Let us give the simplest non-trivial example, i.e. the case $\mu = -1$. We have

$$\tilde{v} = -v - \frac{2(2v^2+t)(2v'+2v^2+t)^2 - 8v(2v'+2v^2+t) + 4}{(2v'+2v^2+t)((2v'+2v^2+t)^2(2v'-2v^2-t) + 8v(2v'+2v^2+t) - 4)}. \tag{10}$$

Note that the rational solution $v = 1/t$ is invariant.

One more remark is necessary here. As we explained above, we can, using (9), relate the solution of P_{II} in the 'linearizable' case to that of a P_{II} in a 'rational-solution' case. However, this relation only applies subject to some constraint to the linearizable and rational solutions themselves. In the case $\mu = -\frac{1}{2}$, we have a solution $v = A'/A$ where A is a solution of the Airy equation $A'' + tA/2 = 0$. Substituting this expression for v into (9a) we find indeed that $u = 0$. Conversely, if we start from $u = 0$ and try to use (9b) we only get the indeterminate form 0/0. This does not mean that *any* solution v is acceptable but only those satisfying (9a), namely the linearized solutions, involving one degree of freedom. Thus some special care is needed in the handling of these special solutions.

Let us now turn to the discrete analogue of the relations we derived above. In this case, we start from the discrete P_{II}:

$$\overline{x} + \underline{x} = \frac{zx + \mu}{1 - x^2} \tag{11}$$

where x stands for $x(n)$, $\overline{x} \equiv x(n+1)$, $\underline{x} \equiv x(n-1)$, and $z \equiv z(n) = \delta n + \beta$, with δ, β and μ constants. We consider the case $\mu = 0$. Multiplying both sides of (11) by x we can introduce the variable $X = x^2$ and the auxiliary variable $Y = x\overline{x}$. Equation (11), with $\mu = 0$, now reads

$$Y + \underline{Y} = \frac{zX}{1 - X} \tag{12a}$$

while we also have

$$X\overline{X} = Y^2. \tag{12b}$$

Solving (12*a*) for X and substituting into (12*b*) we obtain an equation for Y alone:

$$\left(\frac{Y+\overline{Y}}{Y+\overline{Y}+\overline{z}}\right)\left(\frac{Y+\underline{Y}}{Y+\underline{Y}+z}\right) = Y^2. \tag{13}$$

This equation was first identified in [9] as the Miura transformation of the equation we have dubbed the alternate d-P_{II} [10]. The relation between the variables X and Y and that of the alternate-P_{II}, u, is given by

$$Y = uX + \frac{\mu}{2} = \frac{\overline{X}}{u} - \frac{\mu}{2}. \tag{14}$$

In the case $\mu = 0$, equation (14) simplifies to $u^2 = \overline{X}/X$, or

$$u = \frac{\overline{x}}{x}. \tag{15a}$$

The second part of the Miura reads

$$X = x^2 = 1 - \frac{z\underline{u}}{1+u\underline{u}}. \tag{15b}$$

Eliminating between (15*a*) and (15*b*) we obtain for u the equation:

$$\frac{\overline{z}}{1+u\overline{u}} + \frac{z}{1+u\underline{u}} = \frac{1}{u} - u + z \tag{16}$$

which is precisely alt-d-P_{II} with the parameter (which normally appears on the RHS) set to zero, i.e. the value for which alt-d-P_{II} possesses linearizable solutions, in terms of the discrete Airy function. Setting aside the auxiliary variables X, Y, we can interpret (15) as the relation between the solutions of d-P_{II} in the 'rational solution' subcase and that of alt-d-P_{II} in the 'linearizable' subcase. Contrary to the continuum case, this special relation between solutions of the discrete P_{II} is a relation between two *different* discrete systems.

In close parallel with the continuous case, we must point out that (15) is valid for a generic solution, and moreover while x fixes u completely, u fixes x only up to the sign. Again, (15) cannot be applied without precaution to the linearized solution of (16) and the rational solutions of (11). In the former case we have u satifying $1 + u\overline{u} - \overline{z}u = 0$ and (15*b*) implies $x = 0$. Conversely, if we start from the latter solution, (15*a*) reduces to the indeterminate form 0/0 just as in the continuous case.

As expected from what we said in the introduction, both discrete P_{II}'s being of additive type, they are contiguity relations of continuous Painlevé equations. Given this fact, one would expect the special relation (15) obtained above to be the consequence of some analogous relations which holds at the level of continuous Painlevé equations. It turns out that this is indeed the case. In [9] we have presented the detailed list of quadratic relations for Painlevé equations and, among others, we have identified a quadratic relation between solutions of P_V:

$$w'' = (w')^2\left(\frac{1}{2w} + \frac{1}{w-1}\right) - \frac{w'}{t} + \frac{(w-1)^2}{t^2}\left(\alpha w + \frac{\beta}{w}\right) + \gamma\frac{w}{t} + \frac{\delta w(w+1)}{w-1}. \tag{17}$$

The solution $w(t)$ corresponding to $\alpha = -\beta$, $\gamma = 0$ and $\delta \neq 0$, which we can scale to $\delta = -2$ without loss of generality, is related to the solution $v(s)$ corresponding to $\tilde{\delta} = 0$, $\tilde{\gamma} = 1$, $\tilde{\alpha} = 0$, $\tilde{\beta} = -4\alpha$. The precise relation is $v = 4w/(1+w)^2$, with $s = t^2/2$. (Note that $1 - v$ is a perfect square $1 - v = x^{-2}$ with $x = (1+w)/(1-w)$). Now from the theory of the Painlevé equations it is known that a P_V with $\delta = 0$ is just a Miura transformation of P_{III} [12]:

$$u'' = \frac{(u')^2}{u} - \frac{u'}{t} + u^3 + \frac{1}{t}(au^2 + b) - \frac{1}{u}. \tag{18}$$

The relation reads

$$\frac{du}{dt} + \frac{v+1}{v-1}u^2 + 1 - \frac{b+1}{t}u = 0 \tag{19a}$$

$$u = -\frac{s}{v}\frac{dv}{ds} + \frac{a+b+2}{4}v - \frac{a}{2} + \frac{a-b-2}{4v} \tag{19b}$$

where v, considered as a function of s, with $s = t^2/2$, satisfies a P_V with parameters related to that of P_{III} through $\tilde{\alpha} = (a+b+2)^2/32$, $\tilde{\beta} = -(a-b-2)^2/32$, $\tilde{\gamma} = 1$, $\tilde{\delta} = 0$.

In the particular case at hand we have $\tilde{\gamma} = 1$, $\tilde{\delta} = 0$, but also $\tilde{\alpha} = 0$, $\tilde{\beta} = -4\alpha$ where α is the parameter of the original P_V (17). In this case (19) is simplified to

$$\frac{du}{dt} + \frac{v+1}{v-1}u^2 + 1 + \frac{a+1}{t}u = 0 \tag{20a}$$

$$tu = -\frac{t}{2v}\frac{dv}{dt} - \frac{a}{2} + \frac{a}{2v} \tag{20b}$$

where v is expressed in terms of $w(t)$ through $v = 4w/(1+w)^2$. The normalization chosen for this Miura corresponds to $\delta = -2$ for the P_V equation (17) satisfied by $w(t)$, and $u(t)$ satisfies the P_{III} equation (18) with $a = \sqrt{32\alpha}$, $b = -a - 2$.

Note that the latter relation is just such that (18) has linearizable solutions. Indeed, if u satisfies the Riccati

$$\frac{du}{dt} + u^2 + 1 + \frac{a+1}{t}u = 0 \tag{21}$$

it is a solution of (18) with $b = -a-2$. This corresponds to an infinite value of v, i.e. $w = -1$. Thus all the linearizable solutions of P_{III} are related to the elementary solution $w = -1$ of the P_V (17), the conditions $\alpha = -\beta$ and $\gamma = 0$ being just the requirement for this solution to exist.

The relation of the discrete P_{II}'s (11) and (16) to the continuous P_V and P_{III} allow us to understand their special Miura (15) as a consequence of the Miura (20) above. Indeed, the variable x of d-P_{II} is just the one related to the solution w of P_V through $x = (1+w)/(1-w)$, while the variable u of alt-d-P_{II} is precisely that of P_{III}. In particular, the elementary solution $w = -1$ of P_V corresponds to the elementary solution $x = 0$ of the d-P_{II} (11) for $\mu = 0$. As in all the previous cases, the relationship we have established between P_V and P_{III} is one which is valid between generic solutions, and not just for linearized solutions of P_{III} and the elementary solution of P_V.

A final remark is in order here. The relation between P_{III}/P_V and d-P_{II}/alt-d-P_{II} allows one to view the Miura (15) from a different angle. If we start from the u solution of (18) with $b = -a - 2$ and obtain v from (19*a*), we can compute x through the relation $x^2 = (1-v)^{-1}$. This quantity is precisely the one that enters (15*b*). From the contiguity relation between P_{III}/alt-d-P_{II} we have $z = -a/t$. Using this value we can obtain $\underline{u}$ from (15*b*) and verify that it satisfies (18) for $a \to a + 2$ (and $b = -a - 4$). Thus (15*b*), with the proper z, is just the Schlesinger transformation of P_{III}.

Having seen that these special relations between discrete Painlevé equations are the consequences of relations between continuous equations, we can ask whether there exist genuinely discrete analogues without reference to continuous systems whatsoever. In order to find such a relation, we turn to those purely discrete objects, the q-$\mathbb{P}$'s. In [13] we have identified the q-discrete form of $\mathrm{P_{III}}$:

$$\underline{v}v = \frac{\mu\zeta w + \zeta^2}{w(w-1)} \tag{22a}$$

$$\overline{w}w = \frac{\nu\tilde{\zeta} w + \tilde{\zeta}^2}{v(v-1)} \tag{22b}$$

where $\zeta = \zeta_0\lambda^n$, $\tilde{\zeta} = \zeta\sqrt{\lambda}$, and μ, ν and λ are constants. There exists a Miura transformation, from this q-$\mathrm{P_{III}}$ to some other q-equation, of the form

$$X = \frac{w(v + \mu\zeta - vw)}{A\mu\zeta} \tag{23a}$$

$$Y = \frac{vw - \mu\zeta}{Pv} \tag{23b}$$

with $A^2 = \nu\sqrt{\lambda}/\mu$, $P^2 = \mu\nu/\sqrt{\lambda}$. The variables X and Y satisfy

$$\underline{Y}Y = \frac{(X-A)(X-1/A)}{1-ZX} \tag{24a}$$

$$\overline{X}X = (Y-P)(Y-1/P) \tag{24b}$$

with $Z = -A\mu/\zeta$. This system has a $\mathrm{P_V}$ with $\delta = 0$ as its continuous limit, which is obtained through $\lambda = 1+\epsilon$, $Z = \epsilon^2 t$, $Y = 1 - (X + \epsilon t X'/2) + \mathcal{O}(\epsilon^2)$, $A = 1 + \epsilon a$, $P = 1 + \epsilon p$. At the limit $\epsilon \to 0$ we find a non-canonical form of $\mathrm{P_V}$ for X. The homographic transformation $W = X/(X-1)$ leads to a canonical equation $\mathrm{P_V}$ for W of the form (17) with $\alpha = 2a^2$, $\beta = -2p^2$, $\gamma = -2$ and $\delta = 0$.

In the particular case $\mu\nu = \sqrt{\lambda}$, i.e. $P = \pm 1$ (which can be taken as 1 without loss of generality), there exists for (22) a linearizable solution satisfying the discrete Riccati equations: $v(w-1) = \mu\zeta$, $\overline{w}(v-1) = \nu\tilde{\zeta}$, corresponding to $X = 0$, $Y = 1$. For this value of P, we can find a quadratic relation between (24) and another q-$\mathrm{P_V}$. Putting $Y = 1 - x\overline{x}$, $X = x^2$, equation (24b) is automatically satisfied and (24a) becomes a subcase, identified in [11], of the standard q-$\mathrm{P_V}$ [3]:

$$(x\overline{x} - 1)(x\underline{x} - 1) = \frac{(x-b)(x-1/b)(x-c)(x-1/c)}{(1-zxd)(1-zx/d)}. \tag{25}$$

The subcase at hand (where we define $z = \sqrt{-Z}$) is characterized by the constraints $b = -c (= \sqrt{A})$ and $d = i$, which are just the requirement for the existence of the elementary rational solution $x = 0$. (Note that, formally, $x = 0$ is always a solution of (24) whatever the values of its parameters. However, if we expand both sides and subtract 1, we note that an overall factor of x drops out and $x = 0$ is a solution of the resulting equation only when the above constraints are satisfied.) So

$$x^2 = \frac{w(v + \mu\zeta - vw)}{b^2\mu\zeta} \tag{26a}$$

$$x\overline{x} = 1 - w + \frac{\mu\zeta}{v} \tag{26b}$$

is the q-analogue of the relation (20) between a $\mathrm{P_{III}}$, namely (22), in the linearizable case $\mu\nu = 1/\sqrt{\lambda}$, and a $\mathrm{P_V}$, namely (25), in the rational-solution case $c = -b$, $d = \mathrm{i}$. The relation between the single remaining coefficient in each of (22) and (25) is $b^2 = \nu$ while $z^2 = 1/\tilde{\zeta}$.

In this Letter we have explored a new type of special Miura/auto-Bäcklund relations which exist between Painlevé equations. These transformations relate solutions of Painlevé equations for values of the parameters which could not be related through the usual Schlesinger transformations. We have presented these special relations in the case of continuous $\mathbb{P}$'s. For discrete $\mathbb{P}$'s we have shown that these special relations are rather Miura's, relating two different discrete $\mathbb{P}$'s. Finally, as an offshoot of this work we were able to show that it is possible to exhibit a duality among the solutions of P_{II} for the same integer value of the parameter (the rational solution itself being self-dual).

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